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Non-isospectral problem in (2 + 1) dimensions

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Abstract. In this paper the singular manifold method allows us to obtain a non-isospectral Lax pair, Darboux transformations and Miura transformations for an equation in (2+1) dimensions and its modified version. In this way we can iteratively build different kinds of solutions with solitonic behaviour.

1. Introduction

The equation under study in this paper is the following nonlinear equation in (2 + 1):

$$\begin{aligned} 0 &= 4\delta_t + \delta_{xxy} + \sigma_{yy} + 2\delta_x\sigma_{xy} + 2\delta_y^2 \\ 0 &= \sigma_{xx} - \delta_y + \delta_x^2 \end{aligned} \tag{1.1}$$

which can also be written in the form

$$0 = 4\psi_{xt} + \psi_{xxxy} + 4s\psi_{xy}\psi_y + (2\psi_x + 2\psi_{xx}\partial_x^{-1} - s\partial_x^{-1}\partial_y)(2\psi_{xy}\psi_x + s\psi_{yy}) \tag{1.2}$$

by setting $\delta = s\psi$, $s = \pm\mathrm{i}$.

This latter equation (1.2) was proposed by Yu, Toda and Fukuyama [15–17] as the modified version of the equation

$$0 = (4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx}u_y)_x + u_{yyy}. \tag{1.3}$$

It was introduced by Hietarinta *et al* [8] using an extension of Hirota's formalism [7, 18]. In [15], it has been proved that equation (1.3) is a reduction of the KP hierarchy that has the Painlevé property (PP) and admits a Lax representation.

Note that equation (1.3) represents a modification of the Calogero–Bogoyavlenskii–Schif equation [1, 2, 12].

$$0 = 4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx}u_y.$$

Equation (1.3) can be also understood as a modification of the Kadontsev–Petviashili equation. This is why we will refer to (1.3) as the Bogoyavlenskii–Kadontsev–Petviashili (KP–B) equation. Its modified version (1.1) will be called mKP–B. The aim of this paper is to extract a maximum of information about mKP–B by using the singular manifold method (SMM) [13] that, as is well known, is based on the PP [11] for partial differential equations (PDEs) as defined by Weiss *et al* [14]. The plan of this paper is as follows:

- In section 2 we shall analyse equation (1.1) from the point of view of the PP. The leading terms suggest that (1.1) is connected with (1.3) through a *Miura transformation*, which is explicitly constructed. As a bonus, we shall obtain *auto-Bäcklund transformations* for KP–B as well as a *linear superposition* of the solutions of mKP–B in terms of the solutions of KP–B.
- Equation (1.1) has two Painlevé branches but (1.3) has only one. In consequence, it is easier to apply the SMM to (1.3) and, then, use the above linear superposition to obtain solutions of (1.1). In section 3, we apply the SMM to (1.3) and we linearize the singular manifold equations and obtain the Lax pair for KP–B. A new result is that *the Lax pair is in fact non-isospectral*.
- Section 4 is devoted to determining the *Darboux transformations* for KP–B.
- In section 5, the Miura transformation as well as the auto-Bäcklund transformation are used to show that a Painlevé expansion for mKP–B is possible by introducing two singular manifolds.
- We apply the previous results to obtain solitonic solutions of KP–B in section 6. Solutions for mKP–B are also generated.
- The conclusions are presented in section 7.

2. Dominant terms and Miura transformations

2.1. Leading term analysis

In order to perform the Painlevé analysis [14] for equation (1.1), we need to expand the fields σ, δ in a generalized Laurent expansion in terms of an arbitrary singularity manifold $\chi(x, y, t) = 0$. Such an expansion should be of the form [14]

$$\begin{aligned} \delta_x &= \sum_{j=0}^{\infty} a_j(x, y, t)[\chi(x, y, t)]^{j-\alpha} \\ \sigma_x &= \sum_{j=0}^{\infty} b_j(x, y, t)[\chi(x, y, t)]^{j-\beta}. \end{aligned} \tag{2.1}$$

By substituting (2.1) in (1.1), for the leading terms we have

$$\alpha = \beta = \gamma = 1 \qquad a_0 = \pm\chi_x \qquad b_0 = \chi_x \tag{2.2}$$

from which we see that leading term analysis provides two different branches for δ in the Painlevé expansion [4–6]. Checking the PP should be done for both branches. It is not difficult to prove that both branches pass the Painlevé test.

2.2. Miura transformations

If we truncate expansions (2.1) at the constant level, as is required by the SMM [13], we can write the solutions in terms of a singular manifold ϕ, which is not yet an arbitrary function because it is determined by the truncation condition.

Nevertheless, in the present situation the '$\pm$' sign of a_0 indicates that we have two possible singular manifolds: one for the + sign and the other for the − sign. This is why we suggest [4] introducing new fields u and $\hat{u}$ in the form

$$\delta_x = u - \hat{u} \qquad \sigma_x = u + \hat{u} \tag{2.3}$$

in such a way that u and $\hat{u}$, defined as

$$u = \frac{\sigma_x + \delta_x}{2} \qquad \hat{u} = \frac{\sigma_x - \delta_x}{2} \tag{2.4}$$

depend on only one singular manifold [4–6].

By using (1.1), (2.4) can also be written as

$$u_x = \frac{\delta_y - \delta_x^2 + \delta_{xx}}{2} \qquad \hat{u}_x = \frac{\delta_y - \delta_x^2 - \delta_{xx}}{2}. \tag{2.5}$$

It is tedious but not difficult to prove that u and $\hat{u}$ satisfy equation (1.3). Therefore, (2.5) represents the Miura transformations between solutions of (1.1) and (1.3). This is essentially the Miura transformation introduced in [15].

2.3. Auto-Bäcklund transformation

From (2.3), it is immediate to conclude that solutions (δ, σ) of (1.1) can be constructed as linear superpositions of two solutions $(u, \hat{u})$ of (1.3). Nevertheless, these solutions $(u, \hat{u})$ are not independent because from (2.3) and (2.5) we can see that they are related by the auto-Bäcklund transformation:

$$u_x + \hat{u}_x + (u - \hat{u})^2 - \partial_x^{-1}(u_y - \hat{u}_y) = 0. \tag{2.6}$$

In conclusion, *if we have two solutions $(u, \hat{u})$ of (1.3) related by the auto-Bäcklund transformation (2.6), we can construct solutions of (1.1) through (2.3).*

From our point of view, the advantage of such a procedure is that (1.3) has only one Painlevé branch and it is easier to apply the SMM to (1.3) than to (1.1) because, as mentioned above, (1.1) has two branches. With this in mind, the next section will be devoted to applying the SMM to equation (1.3).

3. Singular manifold method for KP–B

3.1. Truncated expansion

Let us return to equation (1.3). To apply the SMM, it is more convenient to write it in the following form:

$$\begin{aligned} 0 &= u_y - \omega_x \\ 0 &= 4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx}u_y + \omega_{yy}. \end{aligned} \tag{3.1}$$

From leading term analysis, it is trivial to see that the truncated Painlevé expansion [13] for u and ω should be

$$\begin{aligned} u' &= u + \frac{\phi_x}{\phi} \\ \omega' &= \omega + \frac{\phi_y}{\phi} \end{aligned} \tag{3.2}$$

where (u, ω) and (u', ω') are solutions of (3.1). Substituting expansion (3.2) in (3.1), we obtain a polynomial in ϕ. If we require that all the coefficients of this polynomial be zero, we obtain the following expressions after some algebraic manipulations (we used Maple V to handle the calculation; the details are in the appendix):

$$u_x = \frac{2p_y - p_x^2 - 2v_x - v^2}{8} + \lambda \tag{3.3}$$

$$u_y = \omega_x = -r - 2\lambda p_x - \frac{v_y + p_x p_y}{2} \tag{3.4}$$

$$\omega_y = rp_x + 2\lambda p_x^2 - p_t - 2\lambda p_y + \frac{p_x^2 p_y - p_{xxy} - vp_{xy} - vp_x p_{xx}}{2} - \frac{p_y^2 + v^2 p_x^2 + p_{xx}^2}{4} \tag{3.5}$$

where v, r and p are defined as

$$v = \frac{\phi_{xx}}{\phi_x} \qquad r = \frac{\phi_t}{\phi_x} \qquad p_x = \frac{\phi_y}{\phi_x} \tag{3.6}$$

and $\lambda = \lambda(y, t)$ is an x-independent function which arises after performing an integration in x (see appendix).

3.2. Singular manifold equations

Furthermore, substitution of (3.3)–(3.5) in (3.1) provides equations to be satisfied by the singular manifold. These equations are (see appendix)

$$0 = [4r + 2p_x p_y + 8\lambda p_x]_x + \left[p_y + 4\lambda - \frac{p_x^2}{2} + v_x - \frac{v^2}{2}\right]_y = 0 \tag{3.7}$$

and the following equation for λ:

$$\lambda_t + 2\lambda\lambda_y = 0. \tag{3.8}$$

It is useful to notice that the compatibility conditions between definitions (3.6) give rise to the following equations:

$$\begin{aligned} \phi_{xxt} = \phi_{txx} &\Longrightarrow v_t = (r_x + vr)_x \\ \phi_{xxy} = \phi_{yxx} &\Longrightarrow v_y = (p_{xx} + vp_x)_x \\ \phi_{yt} = \phi_{ty} &\Longrightarrow p_{xt} = r_y + rp_{xx} - p_x r_x. \end{aligned} \tag{3.9}$$

The set (3.7)–(3.9) are the singular manifold equations.

Note that λ is not necessarily a constant but a function of y and t that is a solution of (3.8).

3.3. Painlevé analysis in singular manifold equations

We can consider singular manifold equations (3.7)–(3.9) as a system of nonlinear coupled PDEs in v and p. This allows us to perform leading terms analysis by setting

$$v \sim v_0 \chi^a \qquad p_x \sim p_0 \chi^b. \tag{3.10}$$

Substitution of (3.10) in (3.7)–(3.9) yields the leading powers

$$a = b = -1 \tag{3.11}$$

and the leading coefficients

$$v_0 = \chi_x \quad p_0 = \pm\chi_x. \tag{3.12}$$

The $\pm$ sign tells us that the Painlevé expansion has two branches. The problem of systems with two Painlevé branches has been discussed in [3–6]. These references suggest that we should consider both branches simultaneously by using two singular manifolds: one for each branch.

3.4. Eigenfunctions and the singular manifold

In agreement with the foregoing, we can write the dominant terms of v and p as

$$\begin{aligned} v &= \frac{\psi_x^+}{\psi^+} + \frac{\psi_x^-}{\psi^-} \\ p_x &= \frac{\psi_x^+}{\psi^+} - \frac{\psi_x^-}{\psi^-} \end{aligned} \tag{3.13}$$

where ψ^+ is the singular manifold for the positive branch and ψ^- for the negative one. We shall later see that ψ^+ and ψ^- are the eigenfunctions of the Lax pair.

By combining (3.6) and (3.13), we obtain the expressions of ϕ in terms of the eigenfunctions

$$\phi_x = \psi^+\psi^- \qquad \phi_y = \psi^-\psi_x^+ - \psi^+\psi_x^-. \tag{3.14}$$

3.5. Linearization of the singular manifold equations: the Lax pair

Substitution of (3.13) in (3.3)–(3.5) gives us the following expressions of u and ω in terms of ψ^+ and ψ^-:

$$u_x = \lambda + \frac{\psi_y^+ - \psi_{xx}^+}{2\psi^+} = \lambda - \frac{\psi_y^- + \psi_{xx}^-}{2\psi^-} \tag{3.15}$$

$$u_{xy} - \omega_y = 2\lambda_y + \frac{2\psi_t^+ + \psi_{yy}^+ + 2u_y\psi_x^+ + 4\lambda\psi_y^+}{\psi^+} \tag{3.16}$$

$$u_{xy} + \omega_y = 2\lambda_y + \frac{2\psi_t^- - \psi_{yy}^- + 2u_y\psi_x^- + 4\lambda\psi_y^-}{\psi^-}. \tag{3.17}$$

Equation (3.15) can be considered to be the spatial part of the Lax pair which, written in a more appropiate way, reads

$$\begin{aligned} 0 &= \psi_{xx}^+ - \psi_y^+ + 2(u_x - \lambda)\psi^+ \\ 0 &= \psi_{xx}^- + \psi_y^- + 2(u_x - \lambda)\psi^-. \end{aligned} \tag{3.18}$$

The temporal part of the Lax pair can be obtained from (3.16), (3.17), which can be written as

$$\begin{aligned} 0 &= \psi_t^+ + \frac{\psi_{yy}^+}{2} - \frac{u_{xy} - \omega_y - 2\lambda_y}{2}\psi^+ + u_y\psi_x^+ + 2\lambda\psi_y^+ \\ 0 &= \psi_t^- - \frac{\psi_{yy}^-}{2} - \frac{u_{xy} + \omega_y - 2\lambda_y}{2}\psi^- + u_y\psi_x^- + 2\lambda\psi_y^-. \end{aligned} \tag{3.19}$$

It is interesting to note that the compatibility condition between (3.18) and (3.19) is equation (3.1) together with condition (3.8). Therefore, (3.18) and (3.19) form the Lax pair for (3.1) and λ is the spectral parameter, even though it is *non-isospectral*. This result generalizes the Lax pair of [15], in which the spectral parameter is absent.

4. Darboux transformations

We can now summarize the above results in the following form:

- Let u and ω be solutions of (3.1) and ϕ_1 a singular manifold for them (associated with a spectral parameter λ_1). This singular manifold can be constructed from two eigenfunctions ψ_1^+ and ψ_1^- as

$$\phi_{1,x} = \psi_1^+\psi_1^- \qquad \phi_{1,y} = \psi_1^-\psi_{1,x}^+ - \psi_1^+\psi_{1,x}^- \tag{4.1}$$

where ψ_1^+ and ψ_1^- are eigenfunctions associated with an eigenvalue λ_1 and therefore satisfying the Lax pairs

$$\begin{aligned} 0 &= \psi_{1,xx}^+ - \psi_{1,y}^+ + 2(u_x - \lambda_1)\psi_1^+ \\ 0 &= \psi_{1,t}^+ + \frac{\psi_{1,yy}^+}{2} - \frac{u_{xy} - \omega_y - 2\lambda_{1,y}}{2}\psi_1^+ + u_y\psi_{1,x}^+ + 2\lambda_1\psi_{1,y}^+ \end{aligned} \tag{4.2}$$

$$\begin{aligned} 0 &= \psi_{1,xx}^- + \psi_{1,y}^- + 2(u_x - \lambda_1)\psi_1^- \\ 0 &= \psi_{1,t}^- - \frac{\psi_{1,yy}^-}{2} - \frac{u_{xy} + \omega_y - 2\lambda_{1,y}}{2}\psi_1^- + u_y\psi_{1,x}^- + 2\lambda_1\psi_{1,y}^- \end{aligned} \tag{4.3}$$

and the spectral parameter λ_1 satisfies

$$\lambda_{1,t} + \lambda_1\lambda_{1,y} = 0. \tag{4.4}$$

- According to (3.2), we can define new solutions u' and ω':

$$\begin{aligned} u' &= u + \frac{\phi_{1,x}}{\phi_1} \\ \omega' &= \omega + \frac{\phi_{1,y}}{\phi_1} \end{aligned} \tag{4.5}$$

whose Lax pairs (associated with the spectral parameter λ_2) will be

$$\begin{aligned} 0 &= \psi_{2,xx}^{'+} - \psi_{2,y}^{'+} + 2(u'_x - \lambda_2)\psi_2^{'+} \\ 0 &= \psi_{2,t}^{'+} + \frac{\psi_{2,yy}^{'+}}{2} - \frac{u'_{xy} - \omega'_y - 2\lambda_{2,y}}{2}\psi_2^{'+} + u'_y\psi_{2,x}^{'+} + 2\lambda_2\psi_{2,y}^{'+} \end{aligned} \tag{4.6}$$

$$\begin{aligned} 0 &= \psi_{2,xx}^{'-} + \psi_{2,y}^{'-} + 2(u'_x - \lambda_2)\psi_2^{'-} \\ 0 &= \psi_{2,t}^{'-} - \frac{\psi_{2,yy}^{'-}}{2} - \frac{u'_{xy} + \omega'_y - 2\lambda_{2,y}}{2}\psi_2^{'-} + u'_y\psi_{2,x}^{'-} + 2\lambda_2\psi_{2,y}^{'-} \end{aligned} \tag{4.7}$$

and we can construct a singular manifold ϕ' for the iterated fields u', ω' through $\psi^{'+}$ and $\psi^{'-}$ as

$$\phi'_{2,x} = \psi_2^{'+}\psi_2^{'-} \qquad \phi'_{2,y} = \psi_2^{'-}\psi_{2,x}^{'+} - \psi_2^{'+}\psi_{2,x}^{'-}. \tag{4.8}$$

4.1. Truncated expansion in the Lax pair

We can consider the Lax pair (4.6) or (4.7) as a system of coupled nonlinear PDEs [5, 9] in $\psi_2^{'+}$, $\psi_2^{'-}$, u' and ω'. Therefore, the SMM can be applied to the Lax pair itself and truncated expansions for $\psi_2^{'+}$ and $\psi_2^{'-}$ should be added to the expansions (4.5). Such expansions can be written as

$$\psi_2^{'+} = \psi_2^+ - \frac{\psi_1^+\Omega^+}{\phi_1} \qquad \psi_2^{'-} = \psi_2^- - \frac{\psi_1^-\Omega^-}{\phi_1}. \tag{4.9}$$

The seminal solutions u, ω, ψ_2^+ and ψ_2^- must satisfy the same Lax pair with the spectral parameter λ_2, which means that

$$\begin{aligned} 0 &= \psi_{2,xx}^+ - \psi_{2,y}^+ + 2(u_x - \lambda_2)\psi_2^+ \\ 0 &= \psi_{2,t}^+ + \frac{\psi_{2,yy}^+}{2} - \frac{u_{xy} - \omega_y - 2\lambda_{2,y}}{2}\psi_2^+ + u_y\psi_{2,x}^+ + 2\lambda_2\psi_{2,y}^+ \end{aligned} \tag{4.10}$$

$$\begin{aligned} 0 &= \psi_{2,xx}^- + \psi_{2,y}^- + 2(u_x - \lambda_2)\psi_2^- \\ 0 &= \psi_{2,t}^- - \frac{\psi_{2,yy}^-}{2} - \frac{u_{xy} + \omega_y - 2\lambda_{2,y}}{2}\psi_2^- + u_y\psi_{2,x}^- + 2\lambda_2\psi_{2,x}^-. \end{aligned} \tag{4.11}$$

Substituting the truncated expansions (4.5) and (4.9) in the Lax pairs (4.6), (4.7) and after some calculation, (we used Maple V for it) we obtain

$$\begin{aligned}\Omega^+ &= \int \psi_2^+ \psi_1^- \,\mathrm{d}x \\ \Omega^- &= \int \psi_1^+ \psi_2^- \,\mathrm{d}x.\end{aligned} \tag{4.12}$$

By substituting (4.1) and (4.12) in (4.5) and (4.9), we can conclude that the set

$$\begin{aligned}u' &= u + \frac{\psi_1^+\psi_1^-}{\int \psi_1^+\psi_1^- \,\mathrm{d}x} \\ \omega' &= \omega + \frac{\psi_{1,x}^+\psi_1^- - \psi_{1,x}^-\psi_1^+}{\int \psi_1^+\psi_1^- \,\mathrm{d}x} \\ \psi_2'^+ &= \psi_2^+ - \frac{\psi_1^+ \int \psi_2^+\psi_1^- \,\mathrm{d}x}{\int \psi_1^+\psi_1^- \,\mathrm{d}x} \\ \psi_2'^- &= \psi_2^- - \frac{\psi_1^- \int \psi_2^-\psi_1^+ \,\mathrm{d}x}{\int \psi_1^+\psi_1^- \,\mathrm{d}x}\end{aligned} \tag{4.13}$$

constitutes a transformation of potentials and eigenfunctions that leaves the Lax pairs invariant. Therefore, (4.13) should be considered as a Darboux transformation [10] for (3.1).

4.2. *Iteration of the singular manifold*

New solutions can be built through the singular manifold and Darboux transformations as follows.

Equation (4.8) can be considered as a nonlinear equation in ϕ_2', $\psi_2'^+$ and $\psi_2'^-$ and it is therefore pertinent to add the following truncated expansion to the set (4.13):

$$\phi_2' = \phi_2 + \frac{\Delta}{\phi_1} \tag{4.14}$$

where ϕ_2 satisfies

$$\phi_{2,x} = \psi_2^+\psi_2^- \qquad \phi_{2,y} = \psi_2^-\psi_{2,x}^+ - \psi_2^+\psi_{2,x}^-. \tag{4.15}$$

Substituting (4.14) and (4.9) in (4.8), one has

$$\Delta = -\Omega^+\Omega^-. \tag{4.16}$$

Since (4.14) defines a singular manifold for u', it can be used to build a new iterated solution:

$$u'' = u' + \frac{\phi_{2,x}'}{\phi_2'}. \tag{4.17}$$

Substitution of equation (4.5) for u' in (4.17) gives

$$u'' = u + \frac{\tau_x}{\tau} \tag{4.18}$$

where

$$\tau = \phi_2'\phi_1 = \phi_1\phi_2 - \Omega^+\Omega^-. \tag{4.19}$$

5. Two singular manifolds for mKP–B

Let us return to equation (1.1), through the superposition (2.3).

Let ϕ and $\hat{\phi}$ be singular manifolds for u and $\hat{u}$ respectively. This means that the truncated Painlevé expansions for u and $\hat{u}$ are

$$\begin{aligned} u' &= u + \frac{\phi_x}{\phi} \qquad & \omega' &= \omega + \frac{\phi_y}{\phi} \\ \hat{u}' &= \hat{u} + \frac{\hat{\phi}_x}{\hat{\phi}} \qquad & \hat{\omega}' &= \hat{\omega} + \frac{\hat{\phi}_y}{\hat{\phi}}. \end{aligned} \tag{5.1}$$

Consequently, according to (2.3), the Painlevé expansion for δ and σ is

$$\begin{aligned} \delta_x' &= u - \hat{u} + \frac{\phi_x}{\phi} - \frac{\hat{\phi}_x}{\hat{\phi}} = \delta_x + \frac{\phi_x}{\phi} - \frac{\hat{\phi}_x}{\hat{\phi}} \\ \sigma_x' &= u + \hat{u} + \frac{\phi_x}{\phi} + \frac{\hat{\phi}_x}{\hat{\phi}} = \sigma_x + \frac{\phi_x}{\phi} + \frac{\hat{\phi}_x}{\hat{\phi}}. \end{aligned} \tag{5.2}$$

Nevertheless, both singular manifolds ϕ and $\hat{\phi}$ are not independent because u and $\hat{u}$ are related by the auto-Bäcklund transformation (2.6)

$$u_x + \hat{u}_x + (u - \hat{u})^2 - \partial_x^{-1}(u_y - \hat{u}_y) = 0. \tag{5.3}$$

The same happens for u' and $\hat{u}'$ which are related as

$$u_x' + \hat{u}_x' + (u' - \hat{u}')^2 - \partial_x^{-1}(u_y' - \hat{u}_y') = 0. \tag{5.4}$$

Substitution of (5.1) in (5.4) provides

$$\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}} = \frac{\phi_x}{\phi}\left(u - \hat{u} + \frac{\phi_{xx}}{2\phi_x} - \frac{\phi_y}{2\phi_x}\right) + \frac{\hat{\phi}_x}{\hat{\phi}}\left(-u + \hat{u} + \frac{\hat{\phi}_{xx}}{2\hat{\phi}_x} + \frac{\hat{\phi}_y}{2\hat{\phi}_x}\right) \tag{5.5}$$

or

$$\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}} = \frac{\phi_x}{\phi}\left(\delta_x + \frac{\phi_{xx}}{2\phi_x} - \frac{\phi_y}{2\phi_x}\right) + \frac{\hat{\phi}_x}{\hat{\phi}}\left(-\delta_x + \frac{\hat{\phi}_{xx}}{2\hat{\phi}_x} + \frac{\hat{\phi}_y}{2\hat{\phi}_x}\right). \tag{5.6}$$

We can conclude that an equation such as mKP–B with two Painlevé branches admits a two singular manifold expansion (5.2) in which both manifolds are related by (5.6).

The Lax pair for mKP–B can be obtained trivially by substituting the Miura transformation in the Lax pair for KP–B.

6. Solutions

In this section, we obtain solutions to KP–B and mKP–B in a systematic way using the previous results. The steps followed in this iterative procedure can be summarized as follows:

(1) We start from seminal solutions of (1.3) and write the Lax pair for them. The solitonic behaviour of the iterated solutions will depend on our choice of the seminal ones.
(2) Solving the Lax pairs, we obtain ψ_1^+, ψ_1^-, ψ_2^+ and ψ_2^-.
(3) We use the results of 2 in (4.1), (4.9), and (4.12) to obtain ϕ_1, ϕ_2, Ω^+ and Ω^-.
(4) We use (4.13) and (4.18) to obtain the first and second iterations u' and u'', respectively.
(5) By combining two solutions for (1.3) we can construct a solution for (1.1) by means of (5.2). The singular manifolds should be related by (5.6).

6.1. Case $u = \omega = 0$

Let us apply the above procedure to the case in which the seminal solutions of (1.3) are

$$u = \omega = 0.$$

If we restrict ourselves to the case in which λ_1 and λ_2 are constant, non-trivial solutions of the Lax pairs (4.2), (4.3) and (4.10), (4.11) are

$$\psi_1^+ = \exp\left[a_1^+ x + ((a_1^+)^2 - 2\lambda_1)y - \left(\frac{(a_1^+)^4 - 4\lambda_1^2}{2}\right)t\right]$$

$$\psi_1^- = \exp\left[a_1^- x - ((a_1^-)^2 - 2\lambda_1)y + \left(\frac{(a_1^-)^4 - 4\lambda_1^2}{2}\right)t\right] \tag{6.1}$$

$$\psi_2^+ = \exp\left[a_2^+ x + ((a_2^+)^2 - 2\lambda_2)y - \left(\frac{(a_2^+)^4 - 4\lambda_2^2}{2}\right)t\right] \tag{6.2}$$

$$\psi_2^- = \exp\left[a_2^- x - ((a_2^-)^2 - 2\lambda_2)y + \left(\frac{(a_2^-)^4 - 4\lambda_2^2}{2}\right)t\right] \tag{6.3}$$

where a_i^+, a_i^- are arbitrary constants. Integration of (4.1) and (4.15) yields

$$\begin{aligned}\phi_1 &= \frac{1}{a_1^+ + a_1^-}(\alpha_1 + \psi_1^+\psi_1^-)\\ \phi_2 &= \frac{1}{a_2^+ + a_2^-}(\alpha_2 + \psi_2^+\psi_2^-)\end{aligned} \tag{6.4}$$

where α_i are arbitrary constants. Using (4.12) one has

$$\begin{aligned}\Omega^+ &= \frac{1}{a_2^+ + a_1^-}(\beta^+ + \psi_2^+\psi_1^-)\\ \Omega^- &= \frac{1}{a_1^+ + a_2^-}(\beta^- + \psi_1^+\psi_2^-)\end{aligned} \tag{6.5}$$

β^+ and β^- are arbitrary constants.

The first iteration provides the solution

$$u' = \partial_x[\ln\phi_1] \qquad \omega' = \partial_y[\ln\phi_1] \tag{6.6}$$

and the second one,

$$u'' = \partial_x[\ln\tau] \qquad \omega'' = \partial_y[\ln\tau] \tag{6.7}$$

$$\begin{aligned}\tau = {} & \frac{1}{(a_1^+ + a_1^-)(a_2^+ + a_2^-)}(\alpha_1\alpha_2 + \alpha_1\psi_2^+\psi_2^- + \alpha_2\psi_1^+\psi_1^- + A_{12}\psi_1^+\psi_1^-\psi_2^+\psi_2^-)\\ & + \frac{1}{(a_2^+ + a_1^-)(a_1^+ + a_2^-)}(\beta^+\beta^- + \beta^+\psi_2^-\psi_1^+ + \beta^-\psi_1^-\psi_2^+)\end{aligned} \tag{6.8}$$

$$A_{12} = \frac{(a_2^+ - a_1^+)(a_2^- - a_1^-)}{(a_2^+ + a_1^-)(a_2^- + a_1^+)}. \tag{6.9}$$

This solution includes the two soliton solutions of [15] when $\beta^+ = \beta^- = 0$.

A particularly interesting case occurs when $a_2^+ = a_1^+$ or $a_2^- = a_1^-$. The interaction term A_{12} therefore vanishes. This case is termed a *resonant state*.

6.2. Case $\delta = \sigma = 0$

In this case the Miura transformation (2.4) gives $u = \hat{u} = \omega = \hat{\omega} = 0$, which means that we can use the solutions of the previous section and construct iterated solutions δ', σ' by using (5.2).

According to (6.1)–(6.4), the solution for ϕ will be

$$\begin{aligned}
\phi &= \frac{1}{a^+ + a^-}(\alpha + \psi^+\psi^-) \\
\psi^+ &= \exp\left[a^+x + ((a^+)^2 - 2\lambda)y - \left(\frac{(a^+)^4 - 4\lambda^2}{2}\right)t\right] \\
\psi^- &= \exp\left[a^-x - ((a^-)^2 - 2\lambda)y + \left(\frac{(a^-)^4 - 4\lambda^2}{2}\right)t\right]
\end{aligned} \tag{6.10}$$

and for $\hat{\phi}$:

$$\begin{aligned}
\hat{\phi} &= \frac{1}{\hat{a}^+ + \hat{a}^-}(\hat{\alpha} + \hat{\psi}^+\hat{\psi}^-) \\
\hat{\psi}^+ &= \exp\left[\hat{a}^+x + ((\hat{a}^+)^2 - 2\lambda)y - \left(\frac{(\hat{a}^+)^4 - 4\hat{\lambda}^2}{2}\right)t\right] \\
\psi^- &= \exp\left[\hat{a}^-x - ((\hat{a}^-)^2 - 2\hat{\lambda})y + \left(\frac{(\hat{a}^-)^4 - 4\hat{\lambda}^2}{2}\right)t\right].
\end{aligned} \tag{6.11}$$

As mentioned before, ϕ and $\hat{\phi}$ should be related by (5.6). By substituting (6.10) and (6.11) in (5.6), we have

$$\hat{\psi}^+\hat{\psi}^- = \psi^+\psi^-$$

and

$$0 = \frac{a^-\hat{\alpha}}{\hat{a}^+ + \hat{a}^-} + \frac{\hat{\alpha}\hat{a}^+}{a^+ + a^-}$$

and consequently

$$\hat{a}^+ = a^+ \qquad \hat{a}^- = a^- \qquad \hat{\lambda} = \lambda \qquad a^-\hat{\alpha} = -\alpha\hat{a}^+.$$

We have for the first iteration

$$\begin{aligned}
\delta' &= \ln\left(\frac{\phi}{\hat{\phi}}\right) \\
\sigma' &= \ln(\phi\hat{\phi})
\end{aligned} \tag{6.12}$$

with

$$\begin{aligned}
\phi &= \frac{1}{a^+ + a^-}(\alpha + \psi^+\psi^-) \\
\hat{\phi} &= \frac{1}{a^+ + a^-}\left(-\frac{a^+}{a^-}\alpha + \psi^+\psi^-\right)
\end{aligned} \tag{6.13}$$

and for the second iteration,

$$\begin{aligned}
\delta'' &= \ln\left(\frac{\tau}{\hat{\tau}}\right) \\
\sigma'' &= \ln(\tau\hat{\tau})
\end{aligned} \tag{6.14}$$

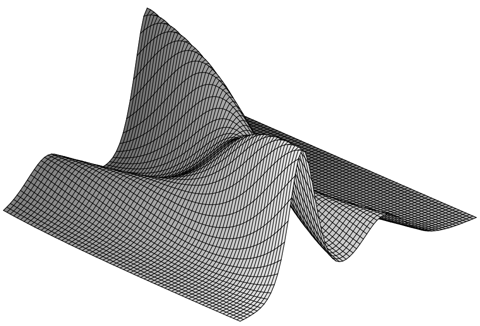

Figure 1. Two-soliton solution δ_x''.

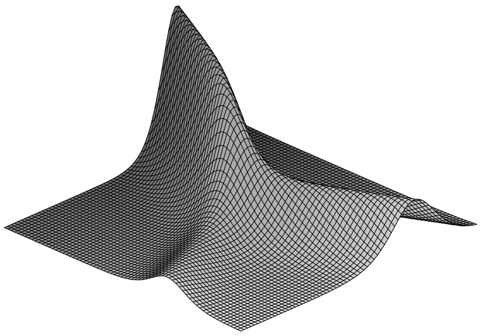

Figure 2. Two-soliton solution δ_x'': resonant case.

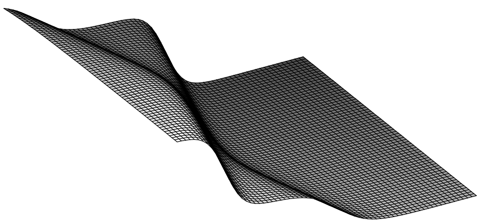

Figure 3. Two-soliton solution for σ_x''.

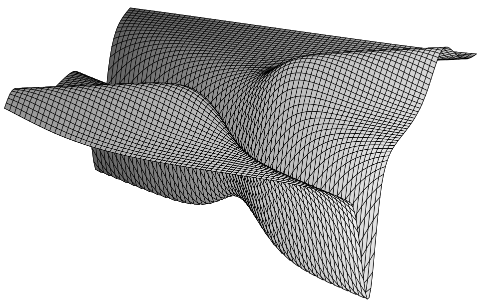

Figure 4. Two-soliton solution for σ_{xy}''.

with

$$\tau = \frac{1}{(a_1^+ + a_1^-)(a_2^+ + a_2^-)}(\alpha_1\alpha_2 + \alpha_1\psi_2^+\psi_2^- + \alpha_2\psi_1^+\psi_1^- + A_{12}\psi_1^+\psi_1^-\psi_2^+\psi_2^-) + \frac{1}{(a_2^+ + a_1^-)(a_1^+ + a_2^-)}(\beta^+\beta^- + \beta^+\psi_2^-\psi_1^+ + \beta^-\psi_1^-\psi_2^+)$$

$$\hat{\tau} = \frac{1}{(a_1^+ + a_1^-)(a_2^+ + a_2^-)}\left(\alpha_1\alpha_2\frac{a_1^+a_2^+}{a_1^-a_2^-} - \alpha_1\frac{a_1^+}{a_1^-}\psi_2^+\psi_2^- - \alpha_2\frac{a_2^+}{a_2^-}\psi_1^+\psi_1^- + A_{12}\psi_1^+\psi_1^-\psi_2^+\psi_2^-\right) + \frac{1}{(a_2^+ + a_1^-)(a_1^+ + a_2^-)}(\hat{\beta}^+\hat{\beta}^- + \hat{\beta}^+\psi_2^-\psi_1^+ + \hat{\beta}^-\psi_1^-\psi_2^+) \tag{6.15}$$

$$A_{12} = \frac{(a_2^+ - a_1^+)(a_2^- - a_1^-)}{(a_2^+ + a_1^-)(a_2^- + a_1^+)}.$$

Figure 1 shows the behaviour of δ_x''. Figure 2 is the resonant case. Figures 3 and 4 are respectively σ_x'' and σ_{xx}''.

7. Conclusions

- An equation in $(2+1)$ dimensions (KP–B) and its modified version (mKP–B) are studied from the point of view of Painlevé analysis.
- Starting with the modified version, we have seen that the equation has two Painlevé branches. The leading term analysis is a very useful indicator of how to construct a Miura transformation between KP–B and mKP–B. It also determines a linear superposition of

the solutions of mKP–B in terms of two solutions of KP–B related by an auto-Bäcklund transformation that is explicitly constructed.

- The Lax pair for KP–B was obtained by performing Painlevé analysis on the singular manifold equations. This allowed us to define the eigenfunctions of the Lax pair.
- In section 4 we considered the Lax pair as a system of coupled PDEs in the fields and eigenfunctions and we obtained the Darboux transformations between two solutions of KP–B. This permitted us to determine an iterative procedure for obtaining solutions from already known ones.
- The linear superposition of the solutions of mKP–B in terms of two solutions of KP–B can be reinterpreted as a Painlevé expansion of the solutions of mKP–B in terms of two singular manifolds. These singular manifolds are not independent. The coupling condition between them can be derived from the auto-Bäcklund transformation for KP–B.
- Finally, we have applied the above results to obtain explicit solutions of both KP–B and mKP–B.

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We would like to thank M J Ablowitz and J A Zagrodzinski for interesting suggestions. This research has been supported in part by DGICYT under project PB95-0947.

Appendix A

Substitution of the truncated expansion (3.2) in (3.1) provides a third degree polynomial in $\frac{1}{\phi}$. Setting each coefficient of each polynomial at zero, we obtain the following equations:

(a) Coefficient in $\frac{1}{\phi^3}$

$$4u_y + 8p_x u_x + 4r + 2v_y + p_x(o_x^2 + 2v_x + v^2) = 0. \tag{A.1}$$

(b) Coefficient in $\frac{1}{\phi^2}$

$$-8u_{xy} - 4p_x u_{xx} 12vu_y + u_x(-24vp_x - 16p_{xx}) - 4v_{xy} - 6vv_y - 8r_x - 12vr \\ +p_x(-3p_{xy} - v_{xx} - 7vv_x - 3v^3 - 3vp_x^2) + p_{xx}(-4v_x - 2v^2 - 3p_x^2) = 0. \tag{A.2}$$

From (A.1), we can obtain u_y as

$$u_y = -\frac{8p_x u_x + 4r + 2v_y + p_x(o_x^2 + 2v_x + v^2)}{4} \tag{A.3}$$

whose substitution in (A.2) gives us

$$8u_{xx} - 2p_{xy} + 2v_{xx} + 2vv_x + 2p_x p_{xx} = 0 \tag{A.4}$$

which can be integrated as

$$8u_x - 2p_y + 2v_x + v^2 + p_x^2 - 8\lambda(y, t) = 0. \tag{A.5}$$

From (A.3) and (A.5), we obtain u_x and u_y as

$$u_x = \frac{2p_y - 2v_x - v^2 - p_x^2}{8} + \lambda \tag{A.6}$$

$$u_y = \omega_x = -r - 2\lambda p_x - \frac{v_y + p_x p_y}{2} \tag{A.7}$$

where $\lambda = \lambda(y, t)$.

The compatibility condition $u_{xy} = u_{yx}$ between (A.6) and (A.7) is precisely the singular manifold equation (3.7):

$$0 = \left[r + \frac{p_x p_y}{2} + 2\lambda p_x \right]_x + \left[\frac{p_y}{4} + \lambda - \frac{p_x^2}{8} + \frac{v_x}{4} - \frac{v^2}{8} \right]_y = 0. \qquad \text{(A.8)}$$

ω_y can be obtained by taking the derivative of (A.7) with respect to y and then performing an integration in x. The result is

$$\omega_y = r p_x + 2\lambda p_x^2 - p_t - 2\lambda p_y + \frac{p_x^2 p_y - p_{xxy} - v p_{xy} - v p_x p_{xx}}{2} - \frac{p_y^2 + v^2 p_x^2 + p_{xx}^2}{4}. \qquad \text{(A.9)}$$

(c) Coefficient in $\frac{1}{\phi}$.

This coefficient is identically zero when (A.6) and (A.7) are substituted.

(d) Coefficient in ϕ^0

This coefficient is obviously equation (3.1). By substituting (A.6), (A.7) and (A.9) in (3.1), we obtain the condition of non-isospectrality:

$$\lambda_t + 2\lambda\lambda_y = 0. \qquad \text{(A.10)}$$

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